Homotopy perturbation method for numerical solutions of coupled Burgers equations with time- and space-fractional derivatives

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Abstract

Purpose – This paper aims to directly extend the homotopy perturbation method to study the coupled Burgers equations with time- and space-fractional derivatives.

Design/methodology/approach – The realistic numerical solutions were obtained in a form of rapidly convergent series with easily computable components.

Findings – The figures show the effectiveness and good accuracy of the proposed method.

Originality/value – The paper obtains realistic numerical solutions in a form of rapidly convergent series with easily computable components. It shows the effectiveness and good accuracy of the proposed method.

Keywords Differential equations, Calculus, Time series analysis

Paper type Research paper

1. Introduction

In recent years, fractional differential equations are stimulated by new examples of applications in fluid mechanics, viscoelasticity, mathematical biology, electrochemistry and physics. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives (He, 1999a), and the fluid-dynamic traffic model with fractional derivatives (He, 1999b) can eliminate the deficiency arising from the assumption of continuum traffic flow. Considerable interest in fractional differential equations has been stimulated due to their numerous applications in the areas of physics and engineering (West *et al.*, 2003). That is because of the fact that, a realistic modeling of a physical phenomenon having dependence not only at the time instant, but also the previous time history can be successfully achieved by using fractional calculus, i.e. the theory of derivatives and integrals of fractional (noninteger) order.

Fractional differential equations have been caught much attention recently due to exact description of nonlinear phenomena. Recent interest mainly covers in analytical approaches to the fractional equations as this paper did, and synchronization for fractional systems as did in (Yu *et al.*, 2009; Sheu *et al.*, 2009; Xu *et al.*, 2008). No analytical method was available before 1998 for such equations, even for linear fractional differential equations. In 1998, the variational iteration method (VIM) was first proposed to solve fractional differential equations with greatest success (He, 1998). Many authors found VIM as an effective way to solving fractional equations, both linear and nonlinear (Odibat and Momani, 2006; Das, 2008). Momani and Odibat (2007) Ganji *et al.* (2008) and Yıldırım (2009a, b) applied the homotopy perturbation method (HPM) to fractional differential equations and revealed that HPM is an alternative analytical method for solving fractional differential equations. Momani *et al.* (2008) and Odibat and Momani (2008) compared solution procedure between VIM and HPM.

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Generally the differential equations are valid for continuous matter, while its fractional partner for discontinuous case, see Equation (6) in (He, 2008c).

The space- and time-fractional coupled Burgers equations have recently been considered by Chen and An (2008) using Adomian decomposition method. The study to coupled Burgers equations is very significant for that the system is a simple model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids, under the effect of gravity (Nee and Duan, 1998). It has been studied by many authors by different methods (Esipov, 1995, Abdou and Soliman, 2005, Dehghan *et al.*, 2007). Especially recently, Dehghan *et al.* (2007) have obtained a good numerical results by using Adomian–Pade technique. However, as we know, the study for the coupled Burgers equations with time- and space-fractional derivatives of this form:

$$\frac{\partial^{\alpha_1} u}{\partial t^{\alpha_1}} = \frac{\partial^2 u}{\partial x^2} + 2u\frac{\partial^{\alpha_2} u}{\partial x^{\alpha_2}} - \frac{\partial(uv)}{\partial x}, \tag{1.1}$$

$$\frac{\partial^{\beta_1} v}{\partial t^{\beta_1}} = \frac{\partial^2 v}{\partial x^2} + 2v\frac{\partial^{\beta_2} v}{\partial x^{\beta_2}} - \frac{\partial(uv)}{\partial x}, \tag{1.2}$$

by the HPM has not been investigated. Here, α_1, α_2 and β_1, β_2 are the parameters standing for the order of the fractional time and space derivatives, respectively, and they satisfy $0 < \alpha_1, \alpha_2, \beta_1, \beta_2 \leq 1$ and $t > 0$. In fact, different response systems can be obtained when at less one of the parameters varies. When $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, the fractional equations reduce to the classical coupled Burgers equation.

The objective of this paper is to extend the application of the HPM to obtain analytic and approximate solutions to the coupled Burgers equations with time- and space-fractional derivatives. The HPM was first proposed by the Chinese mathematician Ji-Huan He (1999b, 2000, 2003, 2004, 2005a, b, 2006a). The essential idea of this method is to introduce a homotopy parameter, say p, which takes values from 0 to 1. When $p = 0$, the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution. As p gradually increases to 1, the system goes through a sequence of deformations, the solution for each of which is close to that at the previous stage of deformation. Eventually at $p = 1$, the system takes the original form of the equation and the final stage of deformation gives the desired solution. One of the most remarkable features of the HPM is that usually just few perturbation terms are sufficient for obtaining a reasonably accurate solution. Considerable research works have been conducted recently in applying this method to a class of linear and nonlinear equations (Yıldırım and Öziş, 2007; Yıldırım, 2008a, b, d; Yıldırım, 2010; Siddiqui *et al.*, 2008; Cveticanin, 2006; Biazar and Ghazvini, 2007, 2008, 2009; Dehghan and Shakeri, 2007, 2008a; Shakeri and Dehghan, 2007, 2008). The interested reader can see the references (He, 2006b, 2006c, 2008a, 2008b; Öziş and Yıldırım, 2007a, b; Yıldırım, 2008c; Yıldırım and Ağırseven, 2009) for last development of HPM. Also references (Dehghan and Shakeri, 2008b, c; Dehghan and Salehi, 2009; Dehghan and Shakeri, 2009, Dehghan and Saadatmandi, 2009; Dehghan and Manafian, 2009; Soltanian *et al.*, 2009; Saadatmandi *et al.*, 2009; Dehghan *et al.*, 2007, 2009) help the readers to know the recent advances in the literature.

The paper is organized as follows. In section 2, some necessary details on the fractional calculus are provided. In section 3, the coupled Burgers equations with time- and space- fractional derivatives are studied with the HPM and the numerical results are graphed to show the efficiency as well as the accuracy of the approximate results achieved. Finally, conclusions are followed.

2. Fractional calculus

We give some basic definitions and theories from the fractional calculus which are used further in this paper.

Definition 2.1. *A real function $f(x), x > 0$, is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$, and it is said to be in the space C_μ^m if and only if $f^{(m)} \in C_\mu, m \in N$.*

Definition 2.2. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as*

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0, \quad x > 0,$$
$$J^0 f(x) = f(x).$$

Properties of the operator J^α can be found in references (Podlubny, 1999; Miller and Ross, 1993; Samko *et al.*, 1993; Oldham and Spanier, 1974), we mention only the following. For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$:

- $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$;
- $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$; and
- $J^\alpha x^\gamma = (\Gamma(\gamma+1))/(\Gamma(\alpha+\gamma+1)) x^{\alpha+\gamma}$.

The Riemann-Liouville derivative has certain disadvantages when trying to model real world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^α proposed by Caputo in his work on the theory of viscoelasticity (Luchko and Gorneflo, 1998).

Definition 2.3. *The fractional derivative $f(x)$ in the Caputo sense is defined as:*

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \tag{2.1}$$

for $m - 1 < \alpha \leq m, \ m \in N, x > 0, f \in C_{-1}^m$.

Also, we need here two of its basic properties.

Lemma 2.1. *If $m - 1 < \alpha \leq m$, $m \in N$ and $f \in C_\mu^m, \mu \geq -1$, then $D^\alpha J^\alpha f(x) = f(x)$, and,*

$$J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0.$$

The Caputo fractional derivatives are considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem. In this paper, we consider the coupled Burgers equations with time- and space-fractional derivatives, and the fractional derivatives are taken in Caputo sense as follows.

Definition 2.4. *For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as:*

$$D_t^{\alpha} u(x,t) = \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial t^m} d\tau, & \text{for } m-1 < \alpha < m \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha = m \in N \end{cases} \tag{2.2}$$

For more information on the mathematical properties of fractional derivatives and integralsone can consult the mentioned references.

3. Numerical examples

3.1 Numerical solutions of the time-fractional coupled Burgers equations

Firstly, we consider the following form of time-fractional coupled Burgers equations (Chen and An, 2008):

$$\begin{aligned} D_t^{\alpha} u &= L_{2x} u + 2u L_x u - L_x(uv) \quad (0 < \alpha \leq 1) \\ D_t^{\beta} v &= L_{2x} v + 2v L_x v - L_x(uv) \quad (0 < \beta \leq 1) \end{aligned} \tag{3.1}$$

with the initial condition:

$$\begin{cases} u(x,0) = f(x) = \sin(x) \\ v(x,0) = g(x) = \sin(x) \end{cases} \tag{3.2}$$

the exact solutions of Equation (3.1) for special case($\alpha = \beta = 1$) is:

$$\begin{cases} u(x,t) = e^{-t} \sin(x) \\ v(x,t) = e^{-t} \sin(x) \end{cases} \tag{3.3}$$

We construct the following homotopy:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} - \frac{\partial^{\alpha} u_0}{\partial t^{\alpha}} = P\left(\frac{\partial^2 u}{\partial x^2} + 2u \frac{\partial u}{\partial x} - \frac{\partial (uv)}{\partial x} - \frac{\partial^{\alpha} u_0}{\partial t^{\alpha}}\right) \tag{3.4}$$

$$\frac{\partial^{\beta} v}{\partial t^{\beta}} - \frac{\partial^{\beta} v_0}{\partial t^{\beta}} = P\left(\frac{\partial^2 v}{\partial x^2} + 2v \frac{\partial v}{\partial x} - \frac{\partial (uv)}{\partial x} - \frac{\partial^{\beta} v_0}{\partial t^{\beta}}\right) \tag{3.5}$$

Assume the solution of Equations (3.4) and (3.5) to be in the form:

$$u = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \cdots\cdots \tag{3.6}$$

$$v = v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \cdots\cdots \tag{3.7}$$

Substituting Equations (3.6) and (3.7) into Equations (3.4) and (3.5) and equating the coefficients of like powers of p, we get following set of differential equations:

$$p^0: \frac{\partial^\alpha u_0}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha} = 0 \tag{3.8}$$

$$p^0: \frac{\partial^\beta v_0}{\partial t^\beta} - \frac{\partial^\beta v_0}{\partial t^\beta} = 0 \tag{3.9}$$

$$p^1: \frac{\partial^\alpha u_1}{\partial t^\alpha} = \frac{\partial^2 u_0}{\partial x^2} + 2u_0 \frac{\partial u_0}{\partial x} - \frac{\partial(u_0 v_0)}{\partial x} \tag{3.10}$$

$$p^1: \frac{\partial^\beta v_1}{\partial t^\beta} = \frac{\partial^2 v_0}{\partial x^2} + 2v_0 \frac{\partial v_0}{\partial x} - \frac{\partial(u_0 v_0)}{\partial x} \tag{3.11}$$

$$p^2: \frac{\partial^\alpha u_2}{\partial t^\alpha} = \frac{\partial^2 u_1}{\partial x^2} + 2u_0 \frac{\partial u_1}{\partial x} + 2u_1 \frac{\partial u_0}{\partial x} - \frac{\partial(u_0 v_1)}{\partial x} - \frac{\partial(u_1 v_0)}{\partial x} \tag{3.12}$$

$$p^2: \frac{\partial^\beta v_2}{\partial t^\beta} = \frac{\partial^2 v_1}{\partial x^2} + 2v_0 \frac{\partial v_1}{\partial x} + 2v_1 \frac{\partial v_0}{\partial x} - \frac{\partial(v_0 u_1)}{\partial x} - \frac{\partial(v_1 u_0)}{\partial x} \tag{3.13}$$

$$\ldots.$$

Solving the above equations, we obtain:

$$u_0 = f(x), \quad v_0 = g(x), \tag{3.14}$$

$$u_1 = f_1(x) \frac{t^\alpha}{\Gamma(\alpha+1)} \tag{3.15}$$

$$v_1 = g_1(x) \frac{t^\beta}{\Gamma(\beta+1)} \tag{3.16}$$

$$u_2 = f_2(x) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + f_3(x) \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \tag{3.17}$$

$$v_2 = g_2(x) \frac{t^{2\beta}}{\Gamma(2\beta+1)} + g_3(x) \frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \tag{3.18}$$

$$\ldots.$$

where,

$$\begin{aligned} f(x) = \sin(x), \quad & f_1(x) = f_{xx} + 2ff_x - fg_x - f_x g, \\ & f_2(x) = f_{1xx} + 2ff_{1x} + 2f_1 f_x - f_{1x} g, \quad f_3(x) = -f_x g_1, \\ g(x) = \sin(x), \quad & g_1(x) = g_{xx} + 2gg_x - gf_x - g_x f, \\ & g_2(x) = g_{1xx} + 2gg_{1x} + 2g_1 g_x - g_{1x} f, \quad g_3(x) = -f_{1x} g, \end{aligned}$$

then we can obtain numerical solutions of time-fractional coupled Burgers Equation (3.1) in series form:

$$u(x,t) = f + f_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + f_3(x)\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \cdots \tag{19}$$

$$v(x,t) = g + g_1(x)\frac{t^{\beta}}{\Gamma(\beta+1)} + g_2(x)\frac{t^{2\beta}}{\Gamma(2\beta+1)} + g_3(x)\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \cdots \tag{20}$$

In order to verify the efficiency and accuracy of the proposed HPM for the time-fractional coupled Burgers equations, we draw figures for the numerical solutions with $\alpha = 0.25, \beta = 0.5, \alpha = 0.75, \beta = 0.9$ as well as the exact solutions (3.3) when $\alpha = \beta = 1$. From the Figures 1 and 2, we can know the series solutions converge rapidly. That's to say a good approximation is achieved by using N-term approximation of the HPM solutions. The accuracy of the numerical solutions obtained depends on how many terms we choose. It is evident that the efficiency of this approach can be dramatically enhanced by computing further terms of *u(x, t)* and *v(x,t)* when the HPM is used.

3.2 Numerical solutions for the space-fractional coupled Burgers equations

Now, we take the space-fractional coupled equations as another example to illustrate the efficiency of the method. As the main method is the same as the above, we will omit the heavy calculation and only give some necessary expressions.

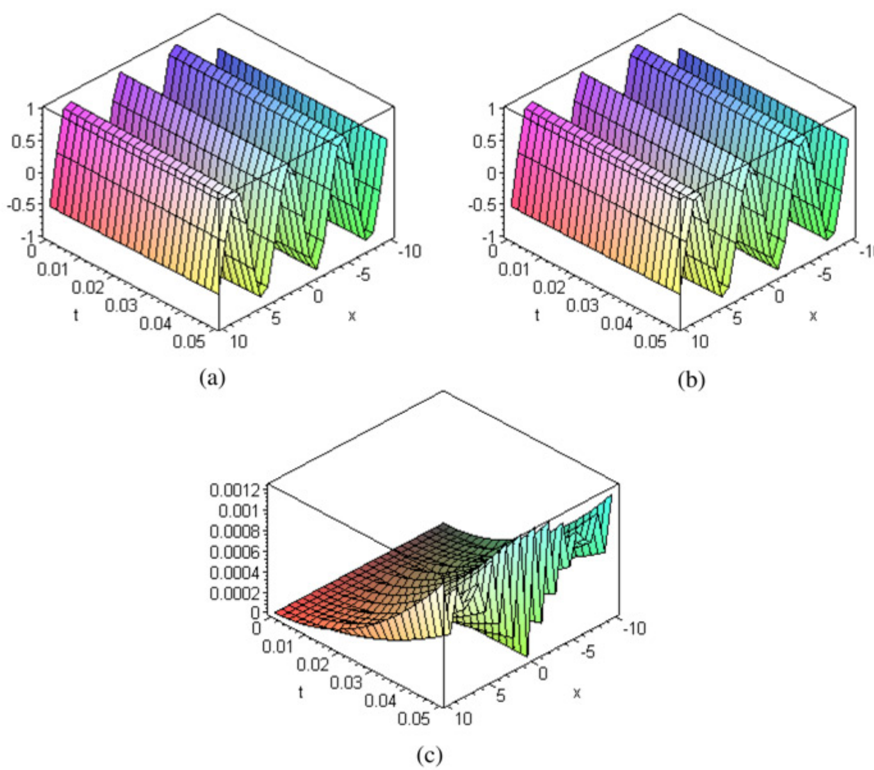


Figure 1.
The surface shows the solution $u(x, t)$ for Equations (1) and (2) when $\alpha = 1$

Notes: (a) Exact solution (3); (b) approximate solution (19); (c) $|u_{ex} - u_{app}|$

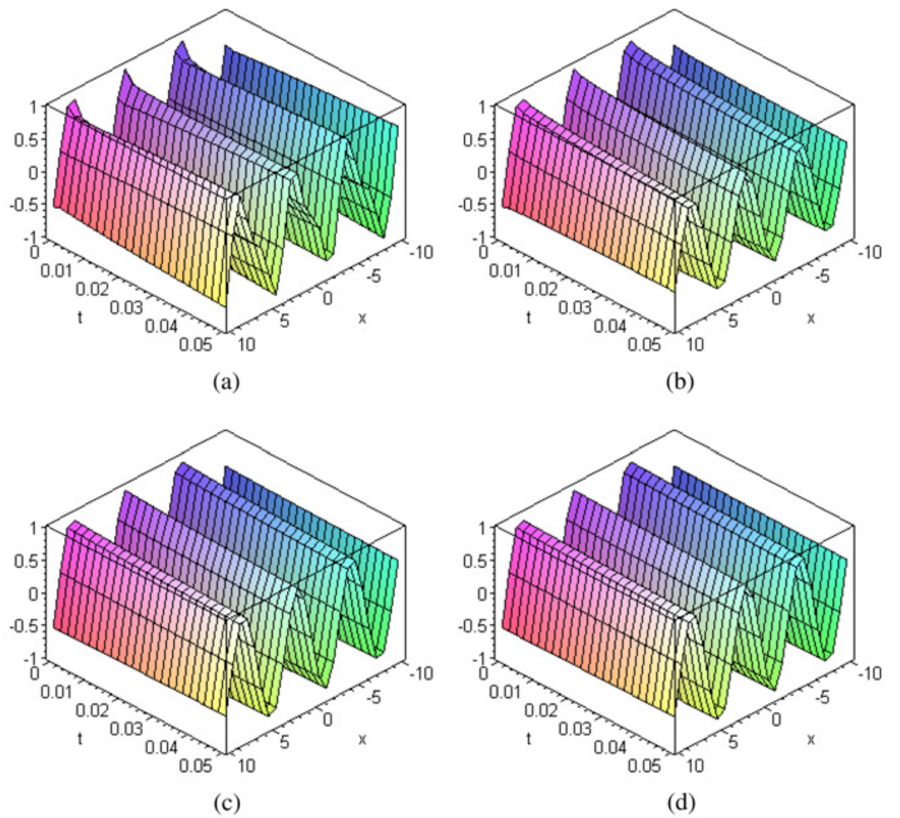


Figure 2. The surface shows the solution $u(x, t)$ and $v(x, t)$ for Equations (1) and (2)

Notes: (a) $\alpha = 0.25$; (b) $\beta = 0.5$; (c) $\alpha = 0.75$; (d) $\beta = 0.9$

We consider the operator form of the space-fractional coupled Burgers equations (Chen and An, 2008):

$$\begin{array}{ll} D_t u = L_{2x} u + 2u D_x^{\alpha} u - L_x(uv) & (0 < \alpha \leq 1) \\ D_t v = L_{2x} v + 2v D_x^{\beta} v - L_x(uv) & (0 < \beta \leq 1) \end{array} \tag{21}$$

with the initial condition:

$$\begin{cases} u(x,0) = f(x) = x^2 \\ v(x,0) = g(x) = x^3 \end{cases} \tag{22}$$

We construct the following homotopy:

$$\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} = p\left(\frac{\partial^2 u}{\partial x^2} + 2u\frac{\partial^{\alpha} u}{\partial x^{\alpha}} - \frac{\partial uv}{\partial x} - \frac{\partial u_0}{\partial t}\right) \tag{23}$$

$$\frac{\partial v}{\partial t} - \frac{\partial v_0}{\partial t} = p\left(\frac{\partial^2 v}{\partial x^2} + 2v\frac{\partial^{\beta} v}{\partial x^{\beta}} - \frac{\partial uv}{\partial x} - \frac{\partial v_0}{\partial t}\right) \tag{24}$$

Substituting Equations (6) and(7) into Equations (23) and (24) and equating the coefficients of like powers of p, we get following set of differential equations:

$$p^0: \frac{\partial u_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0 \tag{25}$$

$$p^0: \frac{\partial v_0}{\partial t} - \frac{\partial v_0}{\partial t} = 0 \tag{26}$$

$$p^1: \frac{\partial u_1}{\partial t} = \frac{\partial^2 u_0}{\partial x^2} + 2u_0 \frac{\partial^\alpha u_0}{\partial x^\alpha} - \frac{\partial(u_0 v_0)}{\partial x} \tag{27}$$

$$p^1: \frac{\partial v_1}{\partial t} = \frac{\partial^2 v_0}{\partial x^2} + 2v_0 \frac{\partial^\alpha v_0}{\partial x^\alpha} - \frac{\partial(u_0 v_0)}{\partial x} \tag{28}$$

$$\cdots$$

Solving the above equations, we obtain:

$$u_0 = x^2, \quad v_0 = x^3, \tag{29}$$

$$u_1 = (2 - 5x^4 + f_1 x^{4-\alpha})t \tag{30}$$

$$v_1 = (6x - 5x^4 + g_1 x^{6-\beta})t \tag{31}$$

$$u_2 = \frac{t^2}{2}\left(f_2 x^{6-2\alpha} + f_3 x^{6-\alpha} + f_4 x^{2-\alpha} + f_5 x^{7-\beta} + 20x^6 + 10x^5 - 72x^2\right) \tag{32}$$

$$v_2 = \frac{t^2}{2}\left(g_2 x^{9-2\beta} + g_3 x^{7-\beta} + g_4 x^{4-\beta} + g_5 x^{6-\alpha} + 20x^6 + 10x^5 - 72x^2\right) \tag{33}$$

where,

$$f(x) = x^2, \quad f_1(x) = \frac{4}{\Gamma(3-\alpha)}, \quad f_2(x) = \left(\frac{4}{\Gamma(3-\alpha)} + \frac{2\Gamma(5-\alpha)}{\Gamma(5-\alpha)}\right) f_1,$$

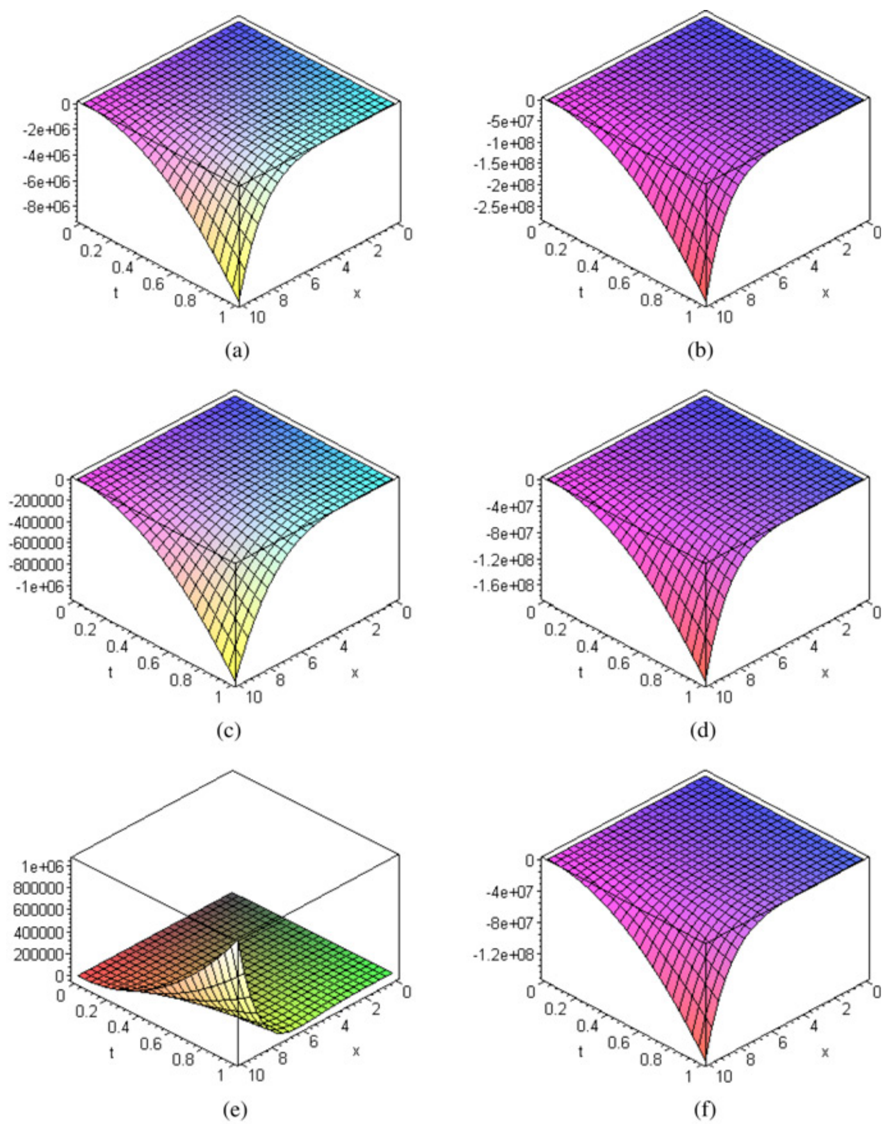
$$f_3(x) = (\alpha - 4) f_1 - \frac{20}{\Gamma(3-\alpha)} - \frac{240}{\Gamma(5-\alpha)}, \quad f_4(x) = (4-\alpha)(3-\alpha) f_1 + \frac{8}{\Gamma(3-\alpha)}, \quad f_5(x) = -2g_1,$$

$$g(x) = x^3, \quad g_1(x) = \frac{12}{\Gamma(4-\beta)}, \quad g_2(x) = \left(\frac{12}{\Gamma(4-\beta)} + \frac{2\Gamma(7-\beta)}{\Gamma(7-\beta)}\right) g_1,$$

$$g_3(x) = -2g_1 - \frac{60}{\Gamma(4-\beta)} - \frac{240}{\Gamma(5-\beta)}, \quad g_4(x) = (6-\beta)(5-\beta) g_1 + \frac{12}{\Gamma(2-\beta)} + \frac{72}{\Gamma(4-\beta)}, \quad g_5(x) = (\alpha - 4) f_1.$$

Then we obtain the numerical solutions of space-fractional Equation (21) in series form:

$$u(x,t) = x^2 + (2 - 5x^4 + f_1 x^{4-\alpha})t + \frac{t^2}{2}\left(f_2 x^{6-2\alpha} + f_3 x^{6-\alpha} + f_4 x^{2-\alpha} + f_5 x^{7-\beta} + 20x^6 + 10x^5 - 72x^2\right) + \cdots \tag{34}$$



Notes: (a) $\alpha = 0.25$; (b) $\beta = 0.5$; (c) $\alpha = 0.75$; (d) $\beta = 0.9$; (e) $\alpha = 1$; (f) $\beta = 1$

Figure 3. The surface shows the solution $u(x, t)$ and $v(x, t)$ for Equations (21) and (22)

$$v(x,t) = x^3 + (6x - 5x^4 + g_1 x^{6-\beta})t + \frac{t^2}{2}\left(g_2 x^{9-2\beta} + g_3 x^{7-\beta} + g_4 x^{4-\beta} + g_5 x^{6-\alpha} + 20x^6 + 10x^5 - 72x^2\right) + \cdots \tag{35}$$

In order to verify the efficiency and accuracy of the proposed HPM for the space-fractional coupled Burgers equations, we draw figures for the numerical solutions with $\alpha = 0.25, \beta = 0.5, \alpha = 0.75, \beta = 0.9$ and $\alpha = \beta = 1$. From the Figure 3, we can know the series solutions converge rapidly.

4. Conclusion

In this paper, by using HPM, we successfully constructed the explicit numerical solutions for the time- and space-fractional coupled Burgers equations with initial condition. We show that the method is straightforward without any restrictive assumptions and special techniques and the continuity of the solution depends on the time- and space-fractional derivatives and the convergent speed is related with terms. Also the HPM is an efficient and powerful method in solving a wide class of equations, in particular, coupled fractional order equations. Generally the results can well explain some unsolved phenomena in porous flow or porous heat problems especially in nano scales, for example, unusual strength, high surface energy, high surface reactivity, high thermal and electric conductivity (Wu *et al.*, 2009; Chen *et al.*, 2009; Mahmood *et al.*, 2008). The main reason is that the porous is discontinuous and can be described using fractional equations. A disadvantage of this new approach is to need an initial value. This technique can not be employed if the problem does not include initial and boundary conditions.

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